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LETTER TO THE EDITOR

Proof of Bose–Einstein condensation for interacting gases with a one-particle spectral gap

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Abstract

Using a specially tuned mean-field Bose gas as a reference system, we establish a positive lower bound on the condensate density for continuous Bose systems with superstable two-body interactions and a finite gap in the one-particle excitations spectrum, i.e. we prove for the first time standard homogeneous Bose–Einstein condensation for such interacting systems.

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1. Introduction

The long-standing problem of proving the existence of a Bose–Einstein condensation (BEC) in non-ideal real Bose gases with a standard two-body interaction has recently attracted interest because of the great success of observing this phenomenon in trapped gases. Notice that much earlier BEC has been observed in liquid ^{4}He, which however has always remained under discussion.

In this letter, we announce our result together with a sketch of the proof about the existence of the standard or zero-mode BEC in Bose gases with realistic superstable two-body interactions and with a gap in the one-particle excitation energy spectrum. To the best of our knowledge, this is the first proof of its sort for homogeneous systems. We are not using any scaling limits (e.g. van der Waals type limits [1–3]) or truncation of particle interactions [4, 5]. We prove that BEC occurs by constructing a positive lower bound for the condensate density which is valid for a low enough temperature and an appropriate large density of particles.

We consider a gas of interacting bosons in cubic boxes $\Lambda = L \times L \times L \subset \mathbb{R}^3$ with periodic boundary conditions. We look here in detail at the three-dimensional case, and comment later

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on the case of other dimensions. We denote by $V = L^3$ the volume of the box Λ. The grand-canonical Hamiltonian of this system reads

$$H^{\Delta}_{\Lambda,g} = T^{\Delta}_{\Lambda} - \mu N_{\Lambda} + gU_{\Lambda} \qquad g > 0 \tag{1}$$

where T^{Δ}_{Λ} is the kinetic energy with gap $\Delta > 0$ in its spectrum

$$T^{\Delta}_{\Lambda} = \sum_{k\in\Lambda^*} \frac{\hbar^2 k^2}{2m} a^{\dagger}_k a_k - \Delta a^{\dagger}_0 a_0. \tag{2}$$

The sum k runs over the set Λ^*, dual to Λ, i.e.

$$\Lambda^* = \{k \in \mathbb{R}^3;\ k_\alpha = 2\pi n_\alpha/L;\ n_\alpha = 0, \pm 1, \ldots;\ \alpha = 1, 2, 3\}.$$

The operators $a^{\dagger}_k$ and a_k are the Bose creation and annihilation operators for mode k. As usual, the mode-occupation number operators are denoted by $N_k = a^{\dagger}_k a_k$, and $N_\Lambda = \sum_{k\in\Lambda^*} N_k$ is the total number operator in Λ.

We assume *a priori* the presence of a gap Δ in the one-particle excitations spectrum, isolating the lowest (zero-mode) energy level. This can be realized by taking appropriate boundary conditions, attractive boundary conditions [6, 7], or such a gap can also be realized by specific inter-particle interactions effectively incorporated in general two-body interactions [5].

Of course μ is the chemical potential and the interaction between the particles is modelled by the two-body interaction term

$$U_\Lambda = \frac{1}{2}\int_{\Lambda^2} \mathrm{d}x\, \mathrm{d}y\, a^{\dagger}(x) a^{\dagger}(y) v(x-y) a(y) a(x) \tag{3}$$

where $a^{\dagger}(x), a^{\dagger}(y)$ and $a(y), a(x)$ are the creation and annihilation operators for the Bose particles at $x, y \in \mathbb{R}^3$. The interaction potential v is assumed to be spherically symmetric, superstable [8], i.e. it satisfies the inequality

$$\sum_{1\leqslant i<j\leqslant n} v(x_i - x_j) \geqslant \frac{A}{2V} n^2 - Bn \tag{4}$$

for some constants $A > 0$, $B \geqslant 0$, and all $n \geqslant 2$, $x_i \in \Lambda$. Consequently, the interaction term (3) satisfies

$$U_\Lambda \geqslant \frac{A}{2V} N^2_\Lambda - BN_\Lambda. \tag{5}$$

This superstability property is, together with the spectral gap (2), the physical foundation of our proof. Intuitively, we might understand that condensation in the ground state (i.e. $k = 0$), which is energetically isolated by a gap Δ, can survive the switching-on of a gentle interaction, and that fluctuations must be of a macroscopical size to overcome this gap and lift particles out of the isolated ground state. More technical parts of the proof are the convexity properties of thermodynamical potentials, such as the pressure, and the use of an optimal choice for the constants A, B in the superstability criterion (4). Indeed, it was proven [8] that continuous L^1-functions of positive type $v : \mathbb{R}^3 \to \mathbb{R}$ are superstable potentials if and only if

$$\hat{v}(0) \geqslant \hat{v}(q) = \int_{\mathbb{R}^3} \mathrm{d}x\, v(x)\, \mathrm{e}^{-\mathrm{i}qx} \geqslant 0 \qquad \forall q \in \mathbb{R}^3 \tag{6}$$

and $\hat{v}(0) > 0$. Moreover, Lewis *et al* [9] proved the existence of the optimal constants $A = \hat{v}(0)(1-\epsilon)$ and $B = v(0)/2$ in equation (4) for this type of potential. Here, $\epsilon > 0$ is an arbitrarily small positive constant. It is related to the size of the system, and can be put to zero after the thermodynamic limit. This optimal choice is of determining importance in our proof of the zero-mode BEC.

2. Sketch of proof

The main idea of the proof is to estimate the Bose condensate of the full model (1) by the condensate of a particularly chosen reference system for which we know the occurrence of condensation. The clever choice of this reference system is the subtle point of our proof. The reference system is a so-called mean-field Bose gas, an exactly solvable model of Bosons [10–16] (for a review, see [5]) defined by the grand-canonical Hamiltonian as

$$H^{\Delta}_{\Lambda,g,\lambda} = T^{\Delta}_{\Lambda} - \mu N_{\Lambda} + g\frac{\lambda}{2V}N^{2}_{\Lambda}. \tag{7}$$

The kinetic energy operator T^{Δ}_{Λ} (2) is as for the general interacting system (1), but the interaction term (3) is replaced by a mean-field interaction term.

The reference system (7), a mean-field Bose gas, emerges as the van der Waals limit of the fully interacting system [1–3]. In that case, the constant λ equals $\hat{v}(0)$ (6), which means that the van der Waals limit reduces the full interaction to the $\hat{v}(0)$ contribution. In our proof we tune the constant λ in order to obtain the best possible lower bound for the condensate density of the full interaction model.

We remark that our reference system (7) does show Bose condensation for large enough densities (i.e. for μ large) at any given temperature. Moreover, systems of the type of our reference system have better properties than the ideal Bose gas which is in many ways a pathological model, e.g. in the sense that there is no equivalence of ensembles [17, 18] and in the sense that the chemical potential is limited by a zero upper bound in order to safeguard thermodynamical stability. These are the reasons for our strategy of using the free of those pathologies reference system (7).

2.1. Thermodynamic properties of the reference system (7)

It is well known that our reference system (7) is a soluble model. In the case of a vanishing gap $\Delta = 0$, the complete solution can be found many times in the literature [5, 10–16]. In particular there is condensation for all dimensions $D \geqslant 3$ at any temperature for large enough densities. It is an exercise for students now to work out the case with gap $\Delta > 0$. There is one main difference with the gapless case, namely the presence of the gap provokes a shift in the chemical potential and its thresholds, and we obtain condensation in all dimensions $D \geqslant 1$. We derive straightforwardly that for the reference model (7) we obtain condensation for all values of the chemical potential satisfying

$$\mu > g\lambda\rho^{P}(\beta, -\Delta) - \Delta$$

where $\rho^{P}(\beta, -\Delta)$ is the total particle density of the perfect Bose gas (PBG) at inverse temperature β and chemical potential equal to $-\Delta$. Moreover, the condensate density

$$\rho^{\Delta}_{0,g,\lambda}(\beta, \mu) = \lim_{V\to\infty}\frac{1}{V}\langle N_0\rangle_{H^{\Delta}_{\Lambda,g,\lambda}}(\beta, \mu)$$

i.e. the particle density at the zero mode in the thermodynamic limit ($V \to \infty$) of the grand-canonical Gibbs states $\langle - \rangle_{H_{\Lambda}}(\beta, \mu)$, in volumes Λ for a certain choice of inverse temperature and chemical potential (β, μ) and Hamiltonian H_{Λ}, is explicitly given by

$$\rho^{\Delta}_{0,g,\lambda} = \frac{\mu + \Delta}{g\lambda} - \rho^{P}(\beta, -\Delta). \tag{8}$$

We also compute the total particle density for given (β, μ) as

$$\rho^{\Delta}_{g,\lambda} = \lim_{V\to\infty}\frac{1}{V}\langle N_{\Lambda}\rangle_{H^{\Delta}_{\Lambda,g,\lambda}} = \frac{\mu + \Delta}{g\lambda}. \tag{9}$$

To prove Bose condensation in the full model (1), we subtract from (3) the long-range part of the interaction proportional to $N_\Lambda^2/2V$. We tune it with a factor, taking into account the optimal stability constants, and add it to the kinetic-energy term. The latter serves as our reference system (7), from which we establish a lower bound on the zero-mode condensate density $\rho_{0,g}^{\Delta}(\beta,\mu) = \lim_{V\to\infty}\langle N_0/V\rangle_{H_{\Lambda,g}^{\Delta}}$ in the fully interacting system (1). In lemma 1, a lower bound on $\rho_{0,g}^{\Delta}(\beta,\mu)$ is given.

Lemma 1. *The zero-mode condensate density $\rho_{0,g}^{\Delta}(\beta,\mu)$ in the thermodynamic limit of grand-canonical Gibbs states of interacting systems (1), with superstable two-body potentials v (6), has the following lower bound:*

$$\rho_{0,g}^{\Delta}(\beta,\mu) \geqslant \frac{\mu}{g\hat{v}(0)} + \frac{g\hat{v}(0)}{2\Delta}(\rho^P(\beta,-\Delta))^2 - \frac{\mu+\Delta}{\Delta}\rho^P(\beta,-\Delta) - \frac{gv(0)}{2\Delta}\rho_g^{(\Delta=0)}(\beta,\mu) - \rho_c^P(\beta). \tag{10}$$

Here $\rho_g^{(\Delta=0)}(\beta,\mu)$ is the total density of the interacting gas without a gap (1). $\rho^P(\beta,-\Delta)$ refers to the total density of the PBG at inverse temperature β and chemical potential $\mu=-\Delta$, and $\rho_c^P(\beta)$ is the critical density of the PBG. The bound is valid for values $\mu > g\hat{v}(0)\rho_c^P(\beta)$, and dimensions $D \geqslant 3$.

Idea of the proof. Using the Bogoliubov convexity inequality [5] we obtain

$$\frac{g}{V}\left\langle W_\Lambda^\lambda\right\rangle_{H_{\Lambda,g}^{\Delta}} \leqslant p_\Lambda\left[H_{\Lambda,g,\lambda}^{\Delta}\right] - p_\Lambda\left[H_{\Lambda,g}^{\Delta}\right] \leqslant \frac{g}{V}\left\langle W_\Lambda^\lambda\right\rangle_{H_{\Lambda,g,\lambda}^{\Delta}}. \tag{11}$$

This gives upper and lower bounds on the difference of the grand-canonical pressure $p_\Lambda[H_\Lambda]$ of the mean-field reference Bose gas (7) and the full model (1). The operator W_Λ^λ is the difference between the interactions of the fully interacting and the mean-field Bose gases, i.e. $W_\Lambda^\lambda = U_\Lambda - \frac{\lambda}{2V}N_\Lambda^2$. The expectation values in (11) can be estimated using, for the lower bound, the superstability properties of the interaction and, for the upper bound, the properties of the equilibrium states of the mean-field reference Bose gas. The lower bound in equation (11) follows from equation (5), and from the tuning of the interaction parameter λ for the mean-field reference Bose gas (7) to the constant A in (5),

$$\frac{g}{V}\left\langle W_\Lambda^A\right\rangle_{H_{\Lambda,g}^{\Delta}} \geqslant -\frac{gB}{V}\langle N_\Lambda\rangle_{H_{\Lambda,g}^{\Delta}}. \tag{12}$$

On the other hand, using the mode by mode gauge invariance of the Gibbs states of the mean-field Bose gas we arrive at the following upper bound for the pressure difference (11),

$$\frac{g}{V}\left\langle W_\Lambda^A\right\rangle_{H_{\Lambda,g,A}^{\Delta}} \leqslant \frac{g}{V^2}\left\langle CN_\Lambda^2 - \frac{\hat{v}(0)}{2}N_0^2\right\rangle_{H_{\Lambda,g,A}^{\Delta}} \tag{13}$$

where $C = \hat{v}(0) - A/2$. It follows from the properties of the mean-field Bose gas that the expectation values in the right-hand side of equation (13) in the limit ($V\to\infty$) are given by $gC\left(\rho_{g,A}^{\Delta}(\beta,\mu)\right)^2 - g\hat{v}(0)\left(\rho_{0,g,A}^{\Delta}(\beta,\mu)\right)^2/2$.

The pressure $p_\Lambda\left[H_\Lambda^{\Delta}\right]$ is an increasing convex function of $\Delta \geqslant 0$. Since the condensate density $\rho_{0,g}^{\Delta}(\beta,\mu)$ is the derivative of the pressure with respect to Δ, by convexity we find for it a lower bound, given by

$$\frac{1}{V}\langle N_0\rangle_{H_{\Lambda,g}^{\Delta}} \geqslant \frac{p_\Lambda\left[H_{\Lambda,g}^{\Delta}\right] - p_\Lambda\left[H_{\Lambda,g}^{(\Delta=0)}\right]}{\Delta}. \tag{14}$$

Analogously, by virtue of the same convexity property, the difference of the pressures between the mean-field Bose gas with and without a gap is bounded from below by the condensate density for the mean-field gas without gap, i.e.

$$\frac{p_\Lambda\big[H^{\Delta}_{\Lambda,g,A}\big] - p_\Lambda\big[H^{(\Delta=0)}_{\Lambda,g,A}\big]}{\Delta} \geqslant \frac{1}{V}\langle N_0\rangle_{H^{(\Delta=0)}_{\Lambda,g,A}}. \tag{15}$$

Adding up these two inequalities and using the Bogoliubov convexity inequality (11) once at $\Delta > 0$ and once at $\Delta = 0$, together with the bounds (12) and (13), we obtain the following lower bound for the condensate density $\rho^{\Delta}_{0,g}(\beta,\mu)$ in the thermodynamic limit ($V \to \infty$):

$$\begin{aligned}\rho^{\Delta}_{0,g}(\beta,\mu) \geqslant{}& \rho^{(\Delta=0)}_{0,g,A}(\beta,\mu) + g\frac{\hat{v}(0)}{2\Delta}\left(\rho^{\Delta}_{0,g,A}(\beta,\mu)\right)^2\\ &-\frac{g}{\Delta}\left(B\rho^{(\Delta=0)}_{g}(\beta,\mu) + C\left(\rho^{\Delta}_{g,A}(\beta,\mu)\right)^2\right).\end{aligned} \tag{16}$$

The lower bound (10) now follows from (16) by use of the explicit expressions for total density and the condensate density of the mean-field Bose gas with a gap (9) and (8), and by the well-known expression for the condensate density in the gapless mean-field model

$$\rho^{(\Delta=0)}_{0,g,\lambda} = \lim_{V\to\infty}\frac{1}{V}\langle N_0\rangle_{H^{(\Delta=0)}_{\Lambda,g,\lambda}} = \frac{\mu}{g\lambda} - \rho^{P}_{c}(\beta)$$

if $\mu > g\lambda\rho^P_c(\beta)$, where $\rho^P_c(\beta)$ is the critical density for the PBG at inverse temperature β. Finally, we need the optimal superstability constants for continuous L^1 potentials of positive type [9], i.e. we take $= \hat{v}(0)$, and $B = v(0)/2$ after the thermodynamic limit, and obtain the expression for the lower bound (10) in the lemma. □

Now we have our main result:

Theorem 2. *Consider a three dimensional system of interacting Bose particles (1), with a superstable two-body potential. There exists a minimal gap* $\Delta_{\min}$ *such that for any finite* $\Delta \geqslant \Delta_{\min}$*, we have* $\rho^{\Delta}_{0,g}(\beta,\mu) > 0$ *or zero-mode condensation.*

Proof. Take any $\eta > 0$, fix a temperature and a chemical potential (β,μ) such that $\mu > g\hat{v}(0)\big(\rho^P_c(\beta) + 3\eta\big)$, then there exists a minimal gap $\Delta_{\min}$ such that for any finite $\Delta \geqslant \Delta_{\min}$,

$$\left|\frac{g\hat{v}(0)}{2\Delta}(\rho^P(\beta,-\Delta))^2 - \frac{gv(0)}{2\Delta}\rho^{(\Delta=0)}_{g}(\beta,\mu) - 2\frac{\mu+\Delta}{\Delta}\rho^P(\beta,-\Delta)\right| < \eta.$$

For these values of the gap Δ we have $\rho^{\Delta}_{0,g}(\beta,\mu) > \eta > 0$, by virtue of the lower bound (10) in lemma 1, and hence we have proven condensation. □

3. Discussion

First, let us remark that our proofs hold without any change in all dimensions $D \geqslant 3$. For $D = 1$ or 2 a similar lower bound on the condensate density can be derived, on the basis of modified convexity arguments (14)–(15), i.e. we have to consider pressure differences of the form $p_\Lambda\big[H^{\Delta}_{\Lambda}\big] - p_\Lambda\big[H^{\Delta_0}_{\Lambda}\big]$, with $0 < \Delta_0 < \Delta$, instead of with $\Delta_0 = 0$ in equations (14)–(15). This yields the substitution of $\rho^{\Delta_0}_{0,g,A}(\beta,\mu)$ and $\rho^{\Delta_0}_{g}(\beta,\mu)$ for $\rho^{(\Delta=0)}_{0,g,A}(\beta,\mu)$ and $\rho^{(\Delta=0)}_{g}(\beta,\mu)$ in equation (16). Hence, also in one- and two-dimensional interacting Bose gases with a gap (1), Bose condensation is proven, in contrast to the Bogoliubov–Hohenberg theorem [19] which yields the absence of BEC for one- or two-dimensional translation invariant continuous Bose systems without a gap.

Finally we want to stress that our results are for homogeneous systems. However, we have to mention here the interesting recent result about condensation for trapped Bose gases [20], i.e. for inhomogeneous systems where a rigorous proof is given of Bose condensation in the so-called Gross–Pitaevskii limit. As such trapped systems always have a gap in the one-particle spectrum, we consider our result as a bridge between homogeneous gapless systems on the one hand and trapped systems on the other hand. Moreover, we hope that our work might be inspiring to establish a proof of BEC for homogeneous systems when the gap tends to zero.

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